M.U 09, 94, ...nce, $\ni e/3 \subseteq T(S_1)$.

M.Sc QNo→ State and Prove open mapping theorem, on Banach Space.

Statement:- If B and B' are Banach spaces, and if T is a continuous linear transformation of B onto B' then T is an open mapping.

Proof:- We must show that if G is an open set in B, then $T(G)$ is an open set in B'. If y is a point in $T(G)$, it is sufficient to produce an open sphere centred on y and contained in $T(G)$. Let x be a point in G such that $Tx = y$. Since

G is open, some $S_r(x) \subseteq G$. But $S_r(x) = x + S_r$. Hence, $x + S_r \subseteq G$. implies that $T(S_r)$ contains some S'_{r_1}. It is clear that $y + S'_{r_1}$ is an open sphere centred on y, and moreover,

$$y + S'_{r_1} \subseteq y + T(S_r) = Tx + T(S_r) = T(x + S_r) \subseteq T(G).$$

Hence, $T(G)$ is an open set.

Thus T is an open mapping.

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Q.No> State and Prove Closed Graph Theorem.

Statement:- If B & B' are Banach spaces and if T is a linear transformation of B into B', then T is continuous iff its graph is closed.

Proof:- Suppose T is continuous. We shall show that the graph of T i.e. $\{(x, Tx) : x \in B\}$ is closed set in $B \times B'$. We note that the metric on the product $B \times B'$ defined by.

$$d((x_1, y_1), (x_2, y_2)) = \max\{\|x_1 - x_2\|, \|y_1 - y_2\|\}$$

determines the product topology on $B \times B'$ and convergence with respect to this metric is equivalent to co-ordinatewise convergence. Let $G = \{(x, Tx) : x \in B\}$. Let (x, y) be a limiting point of G in $B \times B'$. Then there exists a sequence (x_n) of B such that $(x_n, Tx_n) \to (x, y)$. But then $x_n \to x$ and

$Tx_n \to y$. Since T is Continuous, $Tx_n \to Tx$. Hence $y = Tx$. Thus, $(x, y) = (x, Tx) \in G$. Hence G is closed.

Conversely, suppose that the graph G is closed in $B \times B'$. We denote by B_1 the linear space B renormed by $\|x\|_1 = \|x\| + \|Tx\|$. Since,

$$\|Tx\| \leq \|x\| + \|Tx\| = \|x\|_1$$

T is Continuous as mapping of B_1 into B'. It therefore suffices to show that B & B_1 have the same topology. The identity map of B_1 onto B is Continuous for $\|x\| \leq \|x\| + \|Tx\| = \|x\|_1$. If we can show that the B_1 is Complete, then by open mapping theorem the identity mapping will be a homeomorphism of B_1 onto B.

Let (x_1) be a Cauchy seq. in B_1. It follows that x_n and $(T(x_n))$ are Cauchy seq. in B & B'. and since both there spaces are Complete $x_n \to x \in B$ and $T(x_n) \to y \in B'$.

$\therefore (x_n, T(x_n)) \to (x, y)$. our assumption that the graph of T is closed in $B \times B'$ implies that (x, y) lies on this graph, So $T(x) = y$. The Completeness of B_1 now follows from,

$$\|x_n - x\|_1 = \|x_n - x\| + \|T(x_n - x)\|$$
$$= \|x_n - x\| + \|T(x_n) - T(x)\|$$
$$= \|x_n - x\| + \|T(x_n) - y\| \to 0.$$